Revisiting the calculation of inflationary perturbations.[1]

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Abstract

We present a new approximation scheme that allows us to increase the accuracy of analytical predictions of the power spectra of inflationary perturbations for two specific classes of inflationary models. Among these models are chaotic inflation with a monomial potential, power-law inflation and natural inflation (inflation at a maximum). After reviewing the established first order results we calculate the amplitudes and spectral indices for these classes of models at higher orders in the slow-roll parameters for scalar and tensor perturbations.

1 Introduction

Inflationary cosmology [1] is facing exciting times due to a new generation of ground and satellite based experiments to be carried out (e.g., the SDSS, MAP and Planck experiments [2]). Upcoming observations will allow us to determine the values of cosmological parameters with high confidence. To measure the values of these parameters it is necessary to make assumptions on the initial conditions of the density fluctuations that evolved into the observed large scale structure and CMB anisotropies.

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In the simplest inflationary scenario a single scalar field ϕ drives the accelerated expansion of the Universe. Cosmological perturbations are generated by quantum fluctuations of this scalar field and of space-time. Perturbations are adiabatic and gaussian and are characterized by their power spectra. Usually these spectra are described in terms of an amplitude at a pivot scale and the spectral index at this scale. For general models, these quantities are difficult to compute exactly. The state-of-the-art in such calculations are the approximated expressions due to Stewart and Lyth [3], which are obtained up to next-to-leading order in terms of an expansion of the so-called slow-roll parameters. This expansion allows to approximate the solutions to the equations of motion by means of Bessel functions. A recent analysis of Wang et al. [4] shows that the Bessel function approximation cannot be further improved. We will show below that this result depends critically on the assumptions that are made on the relative order of the various slow-roll parameters and that there are two regions in the slow-roll parameter space, where higher accuracy results may be obtained.

To reliably compare analytical predictions with measurements, an error in the theoretical calculations of some percents below the threshold confidence of observations is required. In Ref. [5] it was shown that amplitudes of next-to-leading order power spectra can match the current level of observational precision. However, the error in the spectral index and the resulting net error in the multipole moments of the cosmic microwave background anisotropies might be large due to a long lever arm for wave numbers far away from the pivot scale [6]. A clever choice of the pivot point is essential [6, 7] for todays and future precision measurements. The slow-roll expressions as calculated in [3] are not precise enough for the Planck experiment [6]. Recently, Stewart and Gong [8] presented a new method, also based on the slow-roll expansion, that allows to obtain analytical expressions at higher orders in the slow-roll parameters.

In this paper we show that the Bessel function approximation can be improved to higher orders, without being in conflict with the general argument of [4]. We calculate the amplitudes and indices of the scalar and tensorial perturbations up to higher orders for two specific classes of inflationary models. The first class contains all models that are 'close' to power-law inflation, one example is chaotic inflation with $V \propto \phi^\alpha$. Our second class of models is characterized by extremely slow rolling, an example is inflation near a maximum.

2 The standard formulas

Let us quickly review the derivation and results of the Bessel function approximation. The slow-roll parameters are defined as, [9],

$$\epsilon(\phi) \equiv \frac{2}{\kappa}\left[\frac{H'}{H}\right]^2, \quad \eta(\phi) \equiv \frac{2}{\kappa}\frac{H''}{H}, \quad \xi(\phi) \equiv \frac{2}{\kappa}\left(\frac{H'H'''}{H^2}\right)^{1/2}, \tag{1}$$

with the equations of motion

$$\frac{\dot{\epsilon}}{H} = 2\epsilon(\epsilon - \eta) \,, \quad \frac{\dot{\eta}}{H} = \epsilon\eta - \xi^2 \,. \tag{2}$$

H is the Hubble rate, dot and prime denote derivatives with respect to cosmic time and ϕ, $\kappa = 8\pi/m_{\rm Pl}^2$, and $m_{\rm Pl}$ is the Planck mass. By definition $\epsilon \geq 0$ and it has to be less than unity for inflation to proceed.

2.1 Amplitudes of inflationary perturbations

We call *standard* those formulas which are considered to be the state-of-the-art in the analytical calculation of perturbations spectra, i.e, those obtained by Stewart and Lyth [3].

The general expression for the spectrum of the curvature perturbations is [3]

$$\mathcal{P}_R{}^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}} \left| \frac{u_k}{z} \right| \,. \tag{3}$$

$u_k(\tau)$ are solutions of the mode equation [3, 10]

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2} \right) u_k = 0 \,, \tag{4}$$

where τ is the conformal time and z is defined as $z \equiv a\dot{\phi}/H$. The potential of the mode equation (4) reads [3, 9]

$$\frac{1}{z}\frac{d^2 z}{d\tau^2} = 2a^2H^2 \left(1 + \epsilon - \frac{3}{2}\eta + \epsilon^2 - 2\epsilon\eta + \frac{1}{2}\eta^2 + \frac{1}{2}\xi^2 \right) \,. \tag{5}$$

Despite its appearance as an expansion in slow-roll parameters, Eq. (5) is an exact expression.

The crucial point in the Stewart and Lyth calculations is to use the solution for power-law inflation (where the slow-roll parameters are constant and equal each other) as a pivot expression to look for a general solution in terms of a slow-roll expansion. An answer to whether the slow-roll parameters can be regarded as constants that differ from each other is found it looking at the exact equations of motion (2).

The standard procedure is to consider the slow-roll parameters so small that second order terms in any expression can be neglected. Now aH in (5) can be replaced with help of $(aH)^{-1} \simeq -\tau(1-\epsilon)$. Then, according with Eq. (2) the slow-roll parameters can be fairly regarded as constants and Eq. (4) becomes a Bessel equation readily solved. From that solution the scalar amplitudes are written as [3]

$$\mathcal{P}_R{}^{1/2}(k) = 2^{\nu-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} (1-\epsilon)^{\nu-\frac{1}{2}} \frac{1}{m_{Pl}^2} \left. \frac{H^2}{|H'|} \right|_{k=aH} \,, \tag{6}$$

where ν is given by

$$\nu = \frac{1+\epsilon-\eta}{1-\epsilon} + \frac{1}{2}\,, \tag{7}$$

and k is the wavenumber corresponding to the scale matching the Hubble radius. Expanding solution (6) with ν given by $\nu = 3/2 + 2\epsilon - \eta$ and truncating the results to first order in ϵ and η the standard general expression for the scalar spectrum is obtained,

$$\mathcal{P}_R{}^{1/2} = \frac{\kappa}{4\pi}\left[1-(2C+1)\epsilon + C\eta\right] \frac{H^2}{|H'|}\bigg|_{k=aH}\,, \tag{8}$$

where $C = -2 + \ln 2 + \gamma \simeq -0.73$ is a numerical constant, and γ is the Euler constant that arises when expanding the Gamma function. Eq. (8) is called the next-to-leading order expression for the spectrum amplitudes of scalar perturbations, and from it the leading order is recovered by neglecting first order terms for ϵ and η.

The corresponding equation of motion for the tensorial modes is

$$\frac{d^2 v_k}{d\tau^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{d\tau^2}\right) v_k = 0\,, \tag{9}$$

where,

$$\frac{1}{a}\frac{d^2 a}{d\tau^2} = 2a^2 H^2 \left(1 - \frac{1}{2}\epsilon\right)\,. \tag{10}$$

Neglecting any order of ϵ higher than the first one, Eq. (10) can be written as

$$\frac{1}{a}\frac{d^2 a}{d\tau^2} = \frac{1}{\tau^2}\left(\mu^2 - \frac{1}{4}\right)\,, \tag{11}$$

where,

$$\mu = \frac{1}{1-\epsilon} + \frac{1}{2} \simeq \frac{3}{2} + \epsilon\,. \tag{12}$$

This way, Eq. (9) can be also approximated as a Bessel equation with solution,

$$\mathcal{P}_g^{1/2}(k) = 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}(1-\epsilon)^{\nu-\frac{1}{2}}\ \frac{\sqrt{2\kappa}}{\pi}H\bigg|_{k=aH}\,. \tag{13}$$

Substituting μ given by Eq. (12) in Eq. (13), expanding on ϵ and truncating to first order, the Stewart-Lyth next-to-leading order result is obtained

$$\mathcal{P}_g^{1/2} = \left[1-(C+1)\epsilon\right]\ \frac{\sqrt{2\kappa}}{\pi}H\bigg|_{k=aH}\,. \tag{14}$$

The leading order equation is recovered by neglecting ϵ.

2.2 The spectral indices

To derive the expressions for the spectral indices we introduce here a variation of the standard procedure that has the advantage to allow a careful bookkeeping of the order of slow-roll expressions.

First, let us assume the following ansatz for the power spectrum of general inflationary models,

$$\mathcal{P}_R^{1/2} = h(\epsilon,\eta,\xi) f(\epsilon,\eta,\xi)\,, \tag{15}$$

where $f(\epsilon,\eta,\xi)$ may be written as a Taylor expansion while $h(\epsilon,\eta,\xi)$ is a general function which may be singular or discontinuous at $\epsilon = 0$. Note that any function can be decomposed into this form. We must also consider the following function of the slow-roll parameters:

$$\frac{\kappa}{2}\frac{H}{H'}\frac{d\phi}{d\ln k} = -\frac{1}{1-\epsilon} \equiv g(\epsilon,\eta,\xi)\,. \tag{16}$$

In this expression all of the parameters are to be evaluated at values of ϕ corresponding to $k = aH$. The functions f and g can be expanded in Taylor series,

$$\begin{aligned} f(\epsilon,\eta,\xi) &= a_{00} + a_{10}\epsilon + a_{20}\eta + a_{30}\xi \\ &+ a_{11}\epsilon^2 + a_{12}\epsilon\eta + a_{13}\epsilon\xi + a_{22}\eta^2 + a_{23}\eta\xi + a_{33}\xi^2\cdots\,, \qquad (17)\\ g(\epsilon,\eta,\xi) &= -(1+\epsilon+\epsilon^2+\cdots)\,. \qquad (18)\end{aligned}$$

We proceed with the derivation of the equations for the scalar spectral index

$$n_S(k) - 1 \equiv \frac{d\ln\mathcal{P}_R}{d\ln k}\,. \tag{19}$$

From Eq. (15) we obtain

$$\frac{d\ln\mathcal{P}_R^{1/2}}{d\phi} = \frac{d\ln h(\epsilon,\eta,\xi)}{d\phi} + \frac{d}{d\phi}\left[P(\epsilon,\eta,\xi) - \frac{P^2(\epsilon,\eta,\xi)}{2} + \frac{P^3(\epsilon,\eta,\xi)}{3} - \cdots\right],$$

where

$$P(\epsilon,\eta,\xi) = \tilde{a}_{10}\epsilon + \tilde{a}_{20}\eta + \tilde{a}_{30}\xi + \tilde{a}_{11}\epsilon^2 + \tilde{a}_{12}\epsilon\eta + \tilde{a}_{13}\epsilon\xi + \tilde{a}_{22}\eta^2 + \tilde{a}_{23}\eta\xi + \tilde{a}_{33}\xi^2 + \cdots\,,$$

and $\tilde{a}_{ij} \equiv a_{ij}/a_{00}$. After differentiation,

$$\frac{d\ln\mathcal{P}_R^{1/2}}{d\phi} = \frac{1}{h}\frac{dh}{d\phi} + \left(1 - P(\epsilon,\eta,\xi) + P^2(\epsilon,\eta,\xi) - \cdots\right) P'(\epsilon,\eta,\xi,\epsilon',\eta',\xi')\,, \tag{20}$$

where,

$$P' \equiv \tilde{a}_{10}\epsilon' + \tilde{a}_{20}\eta' + \tilde{a}_{30}\xi' + 2\tilde{a}_{11}\epsilon\epsilon' + \tilde{a}_{12}(\epsilon\eta)' + \tilde{a}_{13}(\epsilon\xi)' + 2\tilde{a}_{22}\eta\eta' + \tilde{a}_{23}(\eta\xi)' + 2\tilde{a}_{33}\xi\xi'$$

plus higher order derivatives. Using Eqs. (16) and (18), and the definitions of the slow-roll parameters (1) we obtain that

$$\frac{n_S - 1}{2} = \left[\sqrt{\frac{2}{\kappa}}\sqrt{\epsilon}\frac{h'}{h} + \left(1 - P + P^2 - \cdots\right)P'\right](1+\epsilon+\epsilon^2+\epsilon^3+\cdots)\,, \tag{21}$$

where P' is written now as,

$$P' = 2\tilde{a}_{10}\epsilon(\epsilon-\eta)+\tilde{a}_{20}(\epsilon\eta-\xi^2)+\tilde{a}_{30}\sqrt{\frac{2}{\kappa}\epsilon}\,\xi'+4\tilde{a}_{11}\epsilon^2(\epsilon-\eta)+\cdots+2\tilde{a}_{22}\eta(\epsilon\eta-\xi^2)\cdots . \tag{22}$$

Expression (21) can be used to any order whenever information on the function h and the coefficients of the expansion (17) is available. The standard result of Stewart and Lyth, Eq. (8), has been tested as a reliable approximation for the scalar spectrum of general inflationary models [5]. Then, it is reasonable to assume

$$h(\epsilon,\eta,\xi) = \frac{\kappa}{4\pi}\left.\frac{H^2}{|H'|}\right|_{k=aH} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{2}}\frac{H}{\sqrt{\epsilon}}\,, \tag{23}$$

$$a_{00} = 1\,, \quad a_{10} = -(2C+1)\,, \quad a_{20} = C\,. \tag{24}$$

With these assumptions, and ignoring the assumptions (constant slow-roll parameters) used to obtain Eq. (8), it is obtained that

$$\sqrt{\frac{2}{\kappa}}\sqrt{\epsilon}\frac{h'}{h} = -2\epsilon+\eta\,, \tag{25}$$

and to second order (the order for which information on the coefficients for (22) is available from Eq. (8)), expression (21) reduces to

$$n_S - 1 = -4\epsilon + 2\eta - 8(C+1)\epsilon^2 + 2(5C+3)\epsilon\eta - 2C\xi^2\,, \tag{26}$$

which is the standard result for the scalar spectral index.

The equation of the tensorial index can be derived in a similar way. By definition

$$n_T(k) \equiv \frac{d\ln\mathcal{P}_g}{d\ln k}\,. \tag{27}$$

The standard equation for tensorial modes is given by Eq. (14). If we now use expression (21) substituting $n_S - 1$ by n_T and

$$h(\epsilon,\eta,\xi) = \frac{\sqrt{2\kappa}}{\pi}H\,, \tag{28}$$

$$a_{00} = 1\,, \quad a_{10} = -(C+1)\,, \tag{29}$$

then we obtain

$$\sqrt{\frac{2}{\kappa}}\sqrt{\epsilon}\frac{h'}{h} = -\epsilon\,, \tag{30}$$

and the standard expression,

$$n_T = -2\epsilon - 2(2C+3)\epsilon^2 + 4(C+1)\epsilon\eta\,. \tag{31}$$

3 Generalizing the Bessel approximation

As can be observed from Eq. (2), the standard assumption used to approximate the slow-roll parameters as constants can not be used beyond the linear term in the slow-roll expansion. Hence, from this point of view, the feasibility of using the Bessel equation to calculate the power spectra is limited to this order. Nevertheless, what it is actually needed is the right hand sides of Eqs. (2) being negligible. That can also be achieved if $|\upsilon| \equiv |\epsilon - \eta| \ll \epsilon$ (we shall call this the power-law approximation) or if $\epsilon \ll |\upsilon|$ (we call this the extreme slow-roll approximation). In both cases we require $\left|\dot{\upsilon} H^{-1}\right| = \left|\xi^2 - \epsilon^2 + 3\epsilon\upsilon\right| \ll \min(\epsilon, |\upsilon|)$. No discrepancy arises with the conclusions in Ref. [4], where

$$\frac{d\ln(\epsilon)}{dN} = -2\upsilon \sim \epsilon$$

is assumed, N being the number of e-foldings. As explained above, this condition is not fulfilled in neither of our approximations.

3.1 Power-law approximation

Power-law inflation is the model which gives rise to the commonly used power-law shape of the primordial spectra, although not always properly implemented [7]. The assumption of a power-law shape of the spectrum has been successful in describing large scale structure from the scales probed by the cosmic microwave background to the scales probed by redshift surveys. It is reasonable to expect that the actual model behind the inflationary perturbations has a strong similarity with power-law inflation. For this class of potentials the precision of the power spectra calculation can be increased while still using the Bessel approximation. If we have a model with $\upsilon \propto \epsilon^n$, then, according with Eq. (2), the first slow-roll parameter can be considered as constant if terms like ϵ^{n+1} and with higher orders are neglected. These considerations, plus the condition $\epsilon^2 \sim \xi^2$, imply the right hand side of the second equation in (2) to be also negligible and, this way, η can be regarded as a constant too. The higher the order in the slow-roll parameters, the smaller should be the difference between them, finally leading, for infinite order in the parameters, to the case of power-law inflation. That means that this approach to the problem of calculating the spectra for more general inflationary models is in fact an expansion around the power-law solution. An example is given by chaotic inflation with the potential $V \propto \phi^\alpha$ with $\alpha > 2$. In this case the slow-roll parameters are given by $\epsilon \simeq \alpha/(4N)$ and $\upsilon \simeq 1/(2N)$ [1], i.e., $\epsilon > \upsilon$. N denotes the number of e-folds of inflation.

Let us proceed with the calculations. In fact all that has to be done is to repeat the calculations of the previous sections keeping the desired order in the slow-roll expansions and neglecting all the terms similar to the right hand sides of Eqs. (2). For example, the next-to-next-to-leading order expression for the scalar power spectrum will be obtained from solution (6) but now with ν given by $\nu = 3/2 + \epsilon + \upsilon + \epsilon^2$. Expanding and keeping terms up to second order, the final expression is

$$\mathcal{P}_R{}^{1/2}(k) = \frac{\kappa}{4\pi}\left[1-(C+1)\epsilon - C\upsilon + (B-1)\epsilon^2\right]\left.\frac{H^2}{|H'|}\right|_{k=aH}, \tag{32}$$

where $B = -2 + C^2/2 + \pi^2/4 \simeq 0.73$. This result is consistent with the result obtained in Ref. [8]. Eq. (8) can be readily recovered from Eq. (32) by neglecting second order terms. For the corresponding expression of the tensorial power spectrum, μ must be $\mu = 3/2 + \epsilon + \epsilon^2$ and,

$$\mathcal{P}_g^{1/2} = \left[1-(C+1)\epsilon + (B-1)\epsilon^2\right]\left.\frac{\sqrt{2\kappa}}{\pi}H\right|_{k=aH}. \tag{33}$$

Equation (14) is obtained from Eq. (33) by neglecting second order terms of ϵ.

3.2 Extreme slow-roll approximation

There is no reason to reject the possibility of an inflationary model with $|\upsilon| \gg \epsilon$. As we shall see later, some important models belong to this class. For these cases, the accuracy of the calculations of the amplitudes can also be increased. We shall focus on the next-to-next-to-leading order. With regards of conditions (2), we neglect terms like ϵ^2, $\epsilon\upsilon$ and $\xi^2-\epsilon^2$ but we keep υ^2 terms. Repeating the calculations under this set of assumptions we obtain for the amplitudes of the scalar spectrum:

$$\mathcal{P}_R{}^{1/2} = \frac{\kappa}{4\pi}\left[1-(C+1)\epsilon - C\upsilon + B\upsilon^2\right]\left.\frac{H^2}{|H'|}\right|_{k=aH}. \tag{34}$$

which again agrees with the result from [8] in the appropriate limit. For tensorial perturbations we note that the Bessel function index (12) does not depend on υ hence, no term like υ^2 will arise at any moment of the calculations. This way, the expression for the tensorial amplitudes is given by Eq. (14).

3.3 The spectral indices

In general, if the approximation neglecting the right hand sides of Eqs. (2) is consistently taken into account, expressions for the spectral indices at any order n are easy to be derived noting that in these cases expression (22) always vanishes and,

$$\frac{n_i}{2} = \left[\sqrt{\frac{2}{\kappa}}\sqrt{\epsilon}\frac{h_i'}{h_i}\right](1+\epsilon+\epsilon^2+\ldots+\epsilon^n), \tag{35}$$

where, n_i is n_S-1 or n_T and h_i is correspondingly given by Eqs.(23) and (28). This way it is obtained

$$\begin{aligned} n_S(k)-1 &= (-4\epsilon+2\eta)(1+\epsilon+\epsilon^2+\ldots+\epsilon^n) \\ &= -2\upsilon - 2\epsilon(1+\epsilon+\epsilon^2+\ldots+\epsilon^n), \qquad (36) \\ n_T(k) &= -2\epsilon(1+\epsilon+\epsilon^2+\ldots+\epsilon^n). \qquad (37) \end{aligned}$$

3.3.1 Relaxing the assumptions

In the same way as the spectral indices (26) and (31) have been derived up to second order in the slow-roll parameters, we can now relax our assumptions on the slow-roll parameters and include higher orders beyond those already included in (36) and (37).

Let us first consider the case of the power-law approximation. The amplitudes are given by Eqs. (32) and (33). We now drop the assumption that ϵ, υ, and ϵ^2 are constant and keep all the contributions from the derivatives of these terms, i.e., $\epsilon\upsilon, \dot{\upsilon}/H, \epsilon^2\upsilon$. However, we have to make sure that no derivatives of terms that have been neglected in the amplitude show up, thus we have to neglect the derivatives of $\epsilon\upsilon, \upsilon^2$, and $\dot{\upsilon}/H$, which gives rise to the conditions $\epsilon\upsilon^2 = -\epsilon\dot{\upsilon}/(2H), \upsilon\dot{\upsilon}/H = 0$, and $[\mathrm{d}(\dot{\upsilon}/H)/\mathrm{d}t]/H = 0$. This means that we are allowed to keep the terms $\epsilon^3, \epsilon^2\upsilon$, and $\epsilon\dot{\upsilon}/H$ beyond the standard expression for the spectral indices.

Consistently with our approach, we must write the expressions for P and P' in terms of our basic parameters ϵ and υ. These expressions are,

$$\begin{aligned} P &= \tilde{a}_{10}\epsilon + \tilde{a}_{20}\upsilon + \tilde{a}_{11}\epsilon^2 + \tilde{a}_{12}\epsilon\upsilon + \tilde{a}_{22}\upsilon^2 + \cdots\,, \qquad (38)\\ P' &= 2\tilde{a}_{10}\epsilon\upsilon + \tilde{a}_{20}\dot{\upsilon}/H \\ &\quad + 4\tilde{a}_{11}\epsilon^2\upsilon + \tilde{a}_{12}\epsilon(2\upsilon^2 + \dot{\upsilon}/H) + 2\tilde{a}_{22}\upsilon\dot{\upsilon}/H + \cdots\,, \qquad (39)\end{aligned}$$

where $\sqrt{2\epsilon/\kappa}\upsilon' = \dot{\upsilon}/H = 2\epsilon^2 - 3\epsilon\eta + \xi^2$. For the scalar contribution in the power-law approximation

$$\tilde{a}_{10} = -(C+1)\,, \quad \tilde{a}_{20} = -C\,, \quad \tilde{a}_{11} = B - 1\,. \qquad (40)$$

Using Eqs. (38) and (39), and taking into account the relaxed approximations discussed above, expression (21) reduces to

$$\begin{aligned} n_S - 1 &= -2\epsilon - 2\upsilon - 2\epsilon^2 - 2(2C+3)\epsilon\upsilon - 2C\dot{\upsilon}/H \\ &\quad -2\epsilon^3 - 2(6C + 17 - \pi^2)\epsilon^2\upsilon - 2C\epsilon\dot{\upsilon}/H\,. \qquad (41)\end{aligned}$$

Correspondingly, for the tensorial index

$$\tilde{a}_{10} = -(C+1)\,, \quad \tilde{a}_{11} = B - 1\,, \qquad (42)$$

and

$$n_T = -2\epsilon - 2\epsilon^2 - 4(C+1)\epsilon\upsilon - 2\epsilon^3 - 2(6C + 16 - \pi^2)\epsilon^2\upsilon\,. \qquad (43)$$

In a similar manner we can derive spectral indices for the extreme slow-roll approximation. Now we have to make use of the conditions $\epsilon^2\upsilon = 0, \epsilon\dot{\upsilon}/H = -2\epsilon\upsilon^2$, and $[\mathrm{d}(\dot{\upsilon}/H)/\mathrm{d}t]/H = 0$. We have now for the scalars

$$\tilde{a}_{10} = -(C+1)\,, \quad \tilde{a}_{20} = -C\,, \quad \tilde{a}_{22} = B\,, \qquad (44)$$

and the spectral index reads,

$$\begin{aligned} n_S - 1 &= -2\epsilon - 2\upsilon - 2\epsilon^2 - 2(2C+3)\epsilon\upsilon - 2C\dot{\upsilon}/H \\ &\quad -2\epsilon^3 + 4C\epsilon\upsilon^2 - (8 - \pi^2)\upsilon\dot{\upsilon}/H\,, \qquad (45)\end{aligned}$$

while the tensorial index is identical to the standard second order slow-roll result,

$$n_T = -2\epsilon - 2\epsilon^2 - 4(C+1)\epsilon\upsilon\,. \tag{46}$$

To conclude this section let us not that our expressions for the spectral indices are in full agreement with the results of Stewart and Gong [8], taking the corresponding approximations into account. This is a quite nontrivial test of our results and of the results of Ref. [8].

4 Testing the expressions

To test all the expressions presented in this paper, we will use the well known exact results for amplitudes and indices calculations in the cases of power-law and natural inflation.

Power-law inflation [11] is an inflationary scenario where,

$$a(t) \propto t^p\,, \quad H(\phi) \propto \exp\left(-\sqrt{\frac{\kappa}{2p}}\,\phi\right), \quad V(\phi) \propto \exp\left(-\sqrt{\frac{2\kappa}{p}}\,\phi\right), \tag{47}$$

with p being a positive constant. It follows from (1) that in this case the slow-roll parameters are constant and equal each other,

$$\epsilon = \eta = \xi = 1/p\,. \tag{48}$$

Thus, this is the limit case of the power-law approximation. Note that condition $\epsilon < 1$ implies $p > 1$.

For this model, the power spectrum of scalar perturbations is given by [12]

$$\mathcal{P}_R{}^{1/2}(k) = \frac{2^{\nu-\frac{1}{2}}}{m_{Pl}^2}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\left(1-\frac{1}{p}\right)^{\nu-\frac{1}{2}}\left.\frac{H^2}{|H'|}\right|_{k=aH}, \tag{49}$$

with

$$\nu \equiv \frac{3}{2} + \frac{1}{p-1}.$$

Expanding up to second order for $p \gg 1$ it is obtained that

$$\mathcal{P}_R{}^{1/2}(k) = \frac{\kappa}{4\pi}\left[1-(C+1)\frac{1}{p}+(B-1)\frac{1}{p^2}\right]\left.\frac{H^2}{|H'|}\right|_{k=aH}, \tag{50}$$

in full correspondence with Eq. (32) when relations (48) are taken into account. Testing the tensorial amplitudes can be easily done by checking that Eqs. (32) and (33) indeed satisfy the relation between amplitudes characteristic of power-law inflation, i.e.,

$$\mathcal{P}_g{}^{1/2}(k) = \frac{4}{\sqrt{p}}\mathcal{P}_R{}^{1/2}(k)\,. \tag{51}$$

For the spectral indices one can see that substitution of relations (48) in Eqs. (36) and (37), as well as in Eqs. (41) and (43), yield

$$\frac{n_S - 1}{2} = \frac{n_T}{2} = -\frac{1}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \frac{1}{1-p}\,, \tag{52}$$

the corresponding relation between indices in power-law inflation.

Another inflationary scenario where precise predictions for the spectra amplitudes and indices can be done is natural inflation [13]. In this case, the potential is given by,

$$V = \Lambda^4 \left[1 \pm \cos(\frac{\phi}{f})\right]\,, \tag{53}$$

where Λ and f are mass scales. For simplicity we choose the plus sign in the remaining calculations. The point here is to analyze inflation near the origin so that the small-angle approximation applies, i.e., $\phi \ll f$. In this case the slow-roll parameters are,

$$\begin{aligned}
\epsilon &= \frac{3}{4}\kappa\left(\sqrt{1+\frac{2}{3}\frac{1}{\kappa f^2}} - 1\right)\phi^2 \simeq 0\,, && (54)\\
\eta &= -\frac{3}{2}\left(\sqrt{1+\frac{2}{3}\frac{1}{\kappa f^2}} - 1\right)\,, && (55)\\
\xi^2 &\simeq 0\,, && (56)\\
\upsilon &\simeq -\eta\,. && (57)
\end{aligned}$$

As it can be observed, in this approximation natural inflation belongs to the class of models that fulfill the conditions of the extreme slow-roll approximation. With the values of the parameters given above, Eq. (6) can be used with the corresponding $\nu = 3/2 + \upsilon$. After expanding and truncating at the proper order

$$\mathcal{P}_R{}^{1/2} = \frac{\kappa}{4\pi}\left[1 - C\upsilon + B\upsilon^2\right]\frac{H^2}{|H'|}\bigg|_{k=aH} \tag{58}$$

is obtained, consistent with Eq. (34). With this same set of assumptions, Eq. (36) and (45) are reduced to

$$n_S - 1 = 2\eta\,, \tag{59}$$

the well known result for natural inflation.

5 Conclusions

In this paper we introduced a simple procedure which allows to improve the precision of the predictions of inflationary perturbations for two specific classes of inflationary models. The method is based on neglecting the evolution in time of the slow-roll parameters ϵ and $\upsilon = \epsilon - \eta$. All monomials of the slow-roll parameters ϵ and υ that are larger than $\max(\dot\epsilon/H, \dot\upsilon/H)$ can be taken into account, the rest is dropped consistently. Two cases arise. In the first one, ϵ is greater than υ. We called this approach power-law approximation. In the opposite case, the value of ϵ is smaller than υ (extreme slow-roll approximation). These approximations plus the standard approximation cover a large space of inflationary models.

We briefly discussed the standard approach to the derivation of the spectral indices. It was noted that the information contained in the expressions for the power spectra amplitudes obtained within the Bessel formalism is used as an approximation to derive the spectral indices for more general models. The obtained expressions have been tested against the exact results for power-law and natural inflation.

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